Thermodynamic implications of some unusual quantum theories.

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Abstract

Various deformations of the position-momentum algebras operators have been proposed. Their implications for single systems like the hydrogen atom or the harmonic oscillator have been addressed. In this paper we investigate the consequences of some of these algebras for macroscopic systems. The key point of our analysis lies in the fact that the modification of the Heisenberg uncertainty relations present in these theories changes the volume of the elementary cell in the hamiltonian phase space and so the measure needed to compute partition functions.

The thermodynamics of a non interacting gas are studied for two members of the Kempf-Mangano-Mann (K.M.M.) deformations. It is shown that the theory which exhibits a minimal uncertainty in length predicts a new behavior at high temperature while the one with a minimal uncertainty in momentum displays unusual features for huge volumes. In the second model negative pressures are obtained and mixing two different gases does not necessarily increase the entropy . This suggests a possible violation of the second law of thermodynamics. Potential consequences of these models in the evolution of the early universe are briefly discussed.

Constructing the Einstein model of a solid for the q deformed oscillator, we find that the subset of eigenstates whose energies are bounded from above leads to a divergent partition function.

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I Introduction

Modern physics is an edifice in which every stone is tightly linked to the others. A slight modification in one area may produce important changes in different fields. Quantum mechanics starts with the commutation relations. Once they have been fixed, using a representation of the Heisenberg algebra we can in principle solve the Schrodinger equation whose solutions potentially contain all the physics of the (non relativistic) system studied. The correspondence principle which states the link between the commutator of two variables of the phase space and their classical Poisson bracket is one of the basic axioms of quantum mechanics. It is not deduced from another assumption and cannot be judged in an isolated manner : only the whole theory can be confronted to experience through its predictions.

Some authors have investigated the consequences of the alterations of these commutation relations on the observable of some physical systems [1] - [11]. In particular, some deformations of the canonical variable commutators studied by Kempf-Mangano-Mann(K.M.M) induce a minimal uncertainty in the position (or the momentum) in a very simple way, providing a toy model with manifest non locality. The implications of some of these quantum structures have been studied for the harmonic oscillator [1, 3] and the hydrogen atom [12]. The transplanckian problem occurring in the usual description of the Hawking mechanism of black hole evaporation has also been addressed in this framework, for the Schwarzchild and the Banados-Teitelboim-Zanelli(B.T.Z) solutions [5, 6].

The purpose of this paper is the investigation of the modifications these models induce, not in the characteristics of a single particle, but in the behavior of a macroscopic system. The modern presentation of thermodynamics basically relies on statistical physics. According to the system under study (isolated, closed or open), one uses an ensemble (microcanonical , canonical or grand canonical) and the corresponding potential (entropy, free energy, grand potential) to derive thermodynamic quantities (pressure, specific heat, chemical potential, etc...).

What has this to do with the commutation relations? The thermodynamic potentials are related to the derivatives of the partition function and the partition function itself is an average (of a quantity which depends on the ensemble used) on the phase space. To define a measure on the phase space, one needs to know the extension of the fundamentals cells. In the wrongly called "classical" statistical Physics, the Heisenberg uncertainty re-

lations are used to show that this volume is the cube of the Planck constant and the indiscernability of particles justifies the Gibbs factor. The difference with quantum statistical physics lies in the fact that in the last case the sum giving the partition function has to be made on a discrete set of states. For example, to explain the specific heat of a solid, the Einstein model treats each atom as a quantized oscillator.

If one modifies the commutators, one changes the energy spectrum and the Heisenberg uncertainty relations. The measure on the phase space is no more the same and this results in new partition functions and consequently different thermodynamic behaviors .

The paper is organized as follows. In the second section we give a brief remind of the basics of statistical physics (to fix the notations)and their application in the derivation of the thermodynamic properties of some simple systems. The third section begins with a short overview of K.M.M. deformations of the momentum-position commutator. The non interacting gas is studied in two special cases: when a minimal uncertainty in position or in momentum is present , in the non relativistic and the ultra relativistic regimes. The Einstein model describing a crystal is also investigated. The last section introduces the harmonic oscillator of the q deformed algebra and uses it to work out the Einstein model of a solid in this particular theory.

II   Basics of statistical physics.

Let us consider, in the usual theory, a system which is in contact with a large heat reservoir and doesn't exchange particles with the surroundings : it has to be studied in the canonical ensemble [13, 14]. Its equilibrium state will be described by a fixed temperature and a fixed particle number while its energy will fluctuate around a mean value. Strictly speaking, for such a system , the particle number N is fixed once and for all. But, one knows that when phase transitions are not present, the descriptions given by the canonical and the grand canonical ensembles are very close. This will be used to compute the chemical potential in the canonical ensemble which is more tractable. Taking for example an assembly of non interacting non relativistic particles in a cubic box of length L, one solves the SchrŠchrodinger equation

$$i\hbar\partial_t\psi = \hat{H}\psi \qquad (2.1)$$

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corresponding to the hamiltonian

$$H = \frac{\vec{p}^2}{2m} \tag{2.2}$$

with the boundary conditions $\psi(0) = \psi(L)$. The energy eigenfunctions are given by

$$\phi_{\vec{k}}(\vec{r}) = \frac{1}{\sqrt{V}} \exp(-iEt) \exp(i\vec{k}.\vec{r}) \tag{2.3}$$

with

$$V = L^3 \;, \; \vec{k} = \frac{2\pi}{L}(n_x, n_y, n_z) \tag{2.4}$$

the n_i being integers. The dispersion relation reads

$$E = \frac{\hbar^2 \vec{k}^2}{2m}. \tag{2.5}$$

The canonical partition function $Z(T, V, N)$ is defined in term of the hamiltonian operator $\hat{H}$ by

$$Z(T, V, N) = Tr \exp\left(-\frac{\hat{H}}{kT}\right). \tag{2.6}$$

Using the complete set of states $\phi_{\vec{k}}(\vec{r})$ displayed earlier, one finds

$$Z(T, V, 1) = \sum_{\vec{k}} \exp(-\frac{\hbar^2}{2mkT}\vec{k}^2). \tag{2.7}$$

The sum on $\vec{k}$ is performed on the discrete set given in Eq(2.4). When the box is sufficiently extended and the temperature not too low, the preceding sum is very well approximated by an integral. Using $Z = Z(T, V, N) = (1/N!)Z(T, V, 1)^N$ one then obtains(with a trivial change of the variable of integration)

$$Z(T, V, N) = \frac{1}{h^{3N}N!} \int d^{3N}p d^{3N}q \exp(-E(q_v, p_v)/kT) \tag{2.8}$$

with $E(p, q) = p^2/2m$. The formula $E(p_v, q_v)$ stands for the energy of the system when it is in a configuration in which the positions and the momenta of the particles are respectively p_v, q_v. Finally

$$Z = \frac{V^N}{N!} \frac{1}{\lambda^{3N}} \qquad \lambda = (\hbar^2/2\pi mkT)^{\frac{1}{2}}. \tag{2.9}$$

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The Gibbs factor $N!$ is present only when the particles are undistinguishable. The free energy is related to the partition function by

$$F(T, V, N) = -kT \ln Z. \tag{2.10}$$

In the variables T, V, N, the pressure P, the entropy S and the chemical potential μ are given by the relations

$$P = -\frac{\partial F}{\partial V} \qquad S = -\frac{\partial F}{\partial T} \qquad \mu = \frac{\partial F}{\partial N} \tag{2.11}$$

while the internal energy is

$$U = F + TS \tag{2.12}$$

and the constant volume specific heat reads

$$C_V = \frac{\partial U}{\partial T}. \tag{2.13}$$

For the case under study, one has

$$\mu = -kT \ln \left\{ \frac{V}{N} \left(\frac{2\pi mkT}{h^2} \right)^{3/2} \right\} \tag{2.14}$$

$$S = Nk \left[\frac{5}{2} + \ln \left\{ \frac{V}{N} \left(\frac{2\pi mkT}{h^2} \right)^{3/2} \right\} \right] \tag{2.15}$$

$$C_V = \frac{3}{2} Nk \tag{2.16}$$

and the equation of state reads

$$PV = NkT. \tag{2.17}$$

The ultrarelativistic gas obeys the dispersion relation $E = c|\vec{p}|$. The corresponding partition function

$$Z_* = \frac{1}{N! h^{3N}} \left(\frac{8\pi k^3 T^3}{c^3} \right)^N V^N. \tag{2.18}$$

leads to the equation of state given in Eq(2.17) and the following expressions for the chemical potential, the entropy and the specific heat

$$\mu = -kT \ln \left[\frac{V}{N} \left(\frac{(8\pi)^{1/3} hc}{kT} \right)^3 \right] \tag{2.19}$$

$$S = Nk \left\{ 4 + \ln \left[\frac{V}{N} \left(\frac{(8\pi)^{1/3} hc}{kT} \right)^3 \right] \right\} \tag{2.20}$$

$$C_V = 3Nk. \tag{2.21}$$

The formula Eq(2.8) admits a semi classical interpretation. The system can classically be symbolized by a point evolving in the phase space. The probability for the system to be in a configuration in which the first particle is in the region $q_1 \pm \Delta q_1, p_1 \pm \Delta p_1$, the second particle in the region $q_2 \pm \Delta q_2, p_2 \pm \Delta p_2, \cdots$ is proportional to $e^{(-E(\vec{q},\vec{p}))}$ and proportional to the number of elementary cells contained in the volume of the aforementioned region. At the quantum level, the Heisenberg uncertainty relation $\Delta p_i \Delta q_i \geq \hbar/2$ assigns to each elementary cell a volume h. The number of such cells contained in the region under consideration is

$$\prod \frac{d^3 p_i d^3 q_i}{\hbar^3} \tag{2.22}$$

which justifies the $\hbar$ factor in Eq.(2.8). The Gibbs factor comes from the symmetrization(antisymmetrization) of wave functions necessary for the description of a multiparticle bosonic(fermionic) state in quantum mechanics. At the semi classical level, it is introduced by hand to avoid an increase in entropy when mixing two formerly separated quantities of the same gas. We shall see in the next section that new commutators change the Heisenberg uncertainty relation and hence the measure needed to compute the partition function.

We shall mostly interest ourselves to two specific cases in this work: the relativistic and the non relativistic non interacting particles. In these two cases,the hamiltonian H solely depends on the momenta. For all the systems we shall consider, we will be interested in the variations of the thermodynamic quantities induced by the uncertainty relations coming from the modified commutators. The new partition functions Z will be related to the usual ones Z_* by a relation of the form

$$Z = Z_* J^N. \tag{2.23}$$

J has to be computed for each modified commutators. The free energy then becomes

$$F = F_* - NkT \ln J. \tag{2.24}$$

The thermodynamic quantities are affected in the following way :

$$P = P_* + NkT \frac{1}{J} \frac{\partial J}{\partial V} \tag{2.25}$$

$$S = S_* + Nk \ln J + NkT \frac{1}{J} \frac{\partial J}{\partial T} \tag{2.26}$$

$$\mu = \mu_* - kT \ln J - NkT \frac{1}{J} \frac{\partial J}{\partial N}. \tag{2.27}$$

$F_*, P_*, \cdots$ are the free energy, the pressure ... obtained from the usual theory, with the unmodified Heisenberg uncertainty. The new internal energy

$$U = U_* + NkT^2 \frac{1}{J} \frac{\partial J}{\partial T} \tag{2.28}$$

leads to the following specific heat at constant volume

$$C_V = C_V^* + 2NkT \frac{1}{J} \frac{\partial J}{\partial T} + NkT^2 \left[-\frac{1}{J^2} \left(\frac{\partial J}{\partial T} \right)^2 + \frac{1}{J} \frac{\partial^2 J}{\partial T^2} \right]. \tag{2.29}$$

The last system we shall be interested in is a crystal. Two approaches are available : the Einstein and the Debye models. They give essentially the same prediction for the specific heat but the last one, which explains the properties of a solid by advocating an exchange of phonons between its sites, gives a better correlation with experiment near zero temperature. We shall nevertheless use the Einstein model because of its simplicity : each site is treated as a quantized harmonic oscillator vibrating at a frequency ω. The partition function is computed exactly

$$Z(T, V, N) = [2 \sinh(\hbar\omega/2kT)]^{-N} \tag{2.30}$$

and leads to the specific heat

$$C_V = Nk \left(\frac{\hbar\omega}{kT} \right)^2 \frac{\exp(\hbar\omega/kT)}{[\exp(\hbar\omega/kT) - 1]^2} \tag{2.31}$$

which varies from zero to $3R$ for a mole as the temperature is increased.

III K.M.M. theory.

The modified commutation relations proposed by Kempf, Mangano and Mann [1] are of the form

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}(1 + \alpha\hat{x}^2 + \beta\hat{p}^2) \tag{3.1}$$

To be more specific, let us consider the case $\alpha = 0$. The other commutators read

$$[\hat{p}_i, \hat{p}_j] = 0$$
$$[\hat{x}_i, \hat{x}_j] = 2i\hbar\beta(\hat{p}_i\hat{x}_j - \hat{p}_j\hat{x}_i) \tag{3.2}$$

The last relation means that we have a "non commutative geometry" since translations along different directions do not commute anymore. The presence of $\vec{p}^2, \vec{x}^2$ in Eq. (3.1) implies that the rotational symmetry, whose generators are

$$\hat{L}_{ij} = \frac{1}{1 + \beta\vec{p}^2}(\hat{x}_i\hat{p}_j - \hat{x}_j\hat{p}_i) \tag{3.3}$$

is still preserved.

One of the remarkable features of this theory is the existence of a minimal length uncertainty $\hbar\sqrt{\beta}$ because of the new Heisenberg inequalities:

$$\Delta x_i \Delta p_j \geq \frac{\hbar}{2}\delta_{ij}\left(1 + \beta\sum_{k=1}^{n}((\Delta\vec{p}_k)^2 + <\vec{p}_k>^2)\right). \tag{3.4}$$

The momentum representation is found to be given by the operators

$$\hat{p}_i.\psi(\vec{p}) = p_i\psi(\vec{p}) \tag{3.5}$$

$$\hat{x}_i.\psi(\vec{p}) = i\hbar(1 + \beta\vec{p}^2)\partial_{p^i}\psi(p) \tag{3.6}$$

acting on a Hilbert space in which the scalar product is given by

$$<\phi|\psi> = \int \frac{d^3\vec{p}}{(1 + \beta\vec{p}^2)}\phi^*(p)\psi(p). \tag{3.7}$$

The presence of a minimal length uncertainty implies that no position representation exists. The concept which proves to be the closest to it is the quasi-position representation in which the operators are non local :

$$\hat{x}_i = \xi_i + \sqrt{\beta}\tan(-\hbar\sqrt{\beta}\partial_{\xi_i}) \tag{3.8}$$

$$\hat{p}_i = \left(\sqrt{\beta^{-1}} \right) \tan(-i\hbar\sqrt{\beta}\partial_{\xi_i}). \tag{3.9}$$

When we refer to a system as a "relativistic" one , this will obviously be linked to its classical behavior. This precision is important because these deformations give rise to theories in which even if the the Poincarré group was a symmetry at the classical level, it is broken at the quantum level.

The constants α, β which appear in Eq(3.1) are free parameters. What numerical values can they assume? Considering the case $\alpha \neq 0$ it was suggested [15] that the minimal length uncertainty $\hbar\sqrt{\beta}$ should be of the order of the Planck scale. We will adopt a more liberal point of view here. The only constraint is that the deformed commutator must reproduce what is predicted by the usual theory and observed experimentally at our energy scale. Our attitude is inspired by recent works which have shown that a new physics may take place well before the Planck scale [16]. As we shall see in the next sections, the behavior of a non interacting gas can be used to derive new, thermodynamic constraints on these parameters. Typically, the new theories will predict new equations of state or (and) new specific heats. The experimental values will be used to obtain bounds on the parameters of the model.

When one introduces the parameters length and momentum scales α, β , it is straightforward that with the Boltzmann constant k , the light velocity c and the mass m of any particle, one can construct on purely dimensional grounds the characteristic temperatures $1/(\beta m k)$ and $c/(k\sqrt{\beta})$ (the first is non relativistic, particle dependent while the second is relativistic and universal) and the critical volume $\alpha^{-3/2}$. We are interested in what happens at these temperatures and volumes.

III-A   The case $\alpha = 0$

III-A.1   The non interacting gas

Let us study a non interacting non relativistic gas. Thinking in terms of quantum mechanics from the start, one has to solve the SchrŠchrodinger equation for a particle in a box. This has to be done in the quasi position representation when $(\alpha = 0)$ because of the lack of a position representation . For simplicity, let us consider the one dimensional case i.e. Eq(2.1): with

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$\hat{p}$ given in Eq. (3.9). One obtains the solution

$$\psi(t, \xi) \div e^{-iEt} \exp\left(\pm\frac{i\xi}{\hbar\sqrt{\beta}} \arctan\sqrt{2m\beta\hbar E)}\right). \tag{3.10}$$

When the boundary condition $\psi(t, \xi = 0) = \psi(t, \xi = L)$ is imposed, one finds the dispersion relation

$$E_n = \frac{1}{2m\beta\hbar} \tan^2\left(\frac{2\pi\hbar\sqrt{\beta}n}{L}\right) \tag{3.11}$$

(n being an integer) already obtained in [1, 5]. This may lead to a cut off in order to avoid an infinite energy. However, this is not obligatory since the energy can become infinite only for very special values of the parameter β.If one does not have a cut-off in the momenta, the energy is no more an increasing function of the quantum number n. If the deformation parameter assumes a special value which forces one to consider such a cut-off, then once the number of particles N is fixed, the energy of the system is bounded.

In the two cases, the integration on the momentum which is given below has an infinite upper bound. The one particle partition function

$$Z(T, V, 1) = \sum_{n=0}^{\infty} e^{-\frac{E_n}{kT}} \tag{3.12}$$

can be approximated by an integral if L is big enough. Introducing the integration variable p by $p^2 = 2mE$ and replacing the length by an integral on position, we find

$$Z(T, V, 1) = \frac{1}{h} \int dx dp \frac{1}{1 + \beta p^2} e^{-p^2/2mkT}. \tag{3.13}$$

One sees that in the new theory, one can keep the habitual dispersion relation and just modify the elementary cell volume. This could be anticipated with a semi classical reasoning. The Heisenberg uncertainty relation derived from the equation $[x, p] = i\hbar(1 + \beta p^2)$ reads

$$\Delta x \Delta p \geq \frac{\hbar}{2}(1 + \beta p^2). \tag{3.14}$$

It assigns to the elementary cells of the phase space of the new theory a volume $\hbar(1+\beta p^2)$ which replaces the usual value $\hbar$. From this we find a simple

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recipe when dealing with the semi classical approximation:it is obtained by keeping the classical dispersion relation but modifying the measure in a way consistent with the Heisenberg uncertainty relation.

Let us now go back to the multi dimensional case. Due to the presence of three dimensional space dimensions, $\hbar$ is replaced by $\hbar^3$. The rotational symmetry results in the replacement of p by $\vec{p}^2$ (This comes from Eq.(2.1))so that the new partition function reads

$$Z = \frac{1}{N!h^{3N}} \int \frac{d^{3N}\vec{p}\, d^{3N}\vec{q}}{(1+\beta\vec{p}^2)^{3N}} \exp(-E(q_v, p_v)/kT). \tag{3.15}$$

III-A.2 The ultra relativistic non interacting gas.

As it is shown in the appendix, the new partition function leads to a non relativistic theory which differs very little from the undeformed case at habitual temperatures. This could be expected since the characteristic temperature T_c is very high for a physically reasonable deformation parameter. The only sector where something new is likely to occur is at temperatures at least equal to T_c. But, at those temperatures, particles are essentially relativistic. With the light velocity, the Boltzmann constant and the parameter β, one can construct, on dimensional grounds, a temperature $T_{cr} = c/(k\sqrt{\beta})$. A new physics will be seen to emerge above this characteristic temperature. Working for simplicity in the ultrarelativistic limit, the expression of J is defined , similarly to the case treated in Appendix $IV.A$ by the integral

$$J = \int_0^\infty \phi(x)dx \tag{3.16}$$

with

$$\phi(x) = 2\pi^{-1/2}e^{-x}x^2 \left(1 + \beta\frac{k^2T^2}{c^2}x^2\right)^{-3} \tag{3.17}$$

with $x = \frac{kT}{c}p$.

As can be seen from Eq.(3.17), J has no volume dependence. This saves the relation $pV = NkT$. The presence of the temperature and the absence of the number of particles in the expression of J results in the fact that the entropy receives two contributing terms (Eq.(2.20)) while the chemical potential last contributing term vanish (Eq.(2.19)). The internal energy remains unchanged is modified and by way of consequence the specific heat at constant volume .

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The dominant contribution to the integral is located near the positive real $\bar{x}$ satisfying

$$-\beta\frac{k^2T^2}{c^2}\bar{x}^3 - 4\beta\frac{k^2T^2}{c^2}\bar{x}^2 - \bar{x} + 2 = 0. \tag{3.18}$$

As explained in the appendix for the non relativistic case, a saddle point approximation is easily computed once when realizes that

$$\phi''(\bar{x}) = -2\pi^{-1/2}e^{-\bar{x}}\left(1 + \beta\frac{k^2T^2}{c^2}\bar{x}^2\right)^{-5}. \tag{3.19}$$

The relevant zone for the saddle point approximation correspond to high temperatures. One then finds , when $T >> T_{cr}$

$$\bar{x} = c/(k\sqrt{\beta}) \tag{3.20}$$

Introducing the universal relativistic characteristic temperature $T_c = c/(kT\sqrt{\beta})$, one finds

$$J = e^{-\frac{T}{T_{cr}}}\left(\frac{T}{T_{cr}}\right)^3 \frac{1}{2^{9/4}\pi^{1/4}} \tag{3.21}$$

$$S = Nk\left\{1 + \ln\left[\frac{V}{N}\left(\frac{(8\pi)^{1/3}hc}{kT}\right)^3\left(\frac{T_{cr}}{T}\right)^3\right]\right\} \tag{3.22}$$

$$\mu = -kT\left\{1 + \ln\left[\frac{V}{N}\left(\frac{(8\pi)^{1/3}hc}{kT}\right)^3\left(\frac{T_{cr}}{T}\right)^3\right]\right\} \tag{3.23}$$

and the equation of state takes the form

$$\rho = 3p\left(1 - \frac{\sqrt{2}}{2}\right). \tag{3.24}$$

It is easily found that an adiabatic process takes the form

$$V = c^{te}NT^6 \tag{3.25}$$

while the law of action of masses reads

$$\frac{X_A^a X_B^b}{X_C^c X_D^d} = c^{te}T^{-6(a+b-c-d)} \tag{3.26}$$

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The dependence of the constants appearing in the last two equations on the characteristic temperature is obvious.

The saddle point approximation used so far gives a picture which, even if it is qualitatively correct, is not the exact result. For example, when $\beta = 0$, Eq(3.21) gives $J = 0.85$ while one knows the correct value to be $J = 1$. The exact result is computed numerically and plotted in the figures. From them one extracts the more accurate values of the σ_i and finds that $\sigma_0 = 1$ as it should. The presence of the V, N in the expression of S makes this theory safe concerning the mixing entropy and the the second law of thermodynamics, contrary to what happens in the case $\beta = 0$ as we shall see in the next section . The equation of state takes the form

$$\rho = 3p(1 + g(T/T_{cr})) \tag{3.27}$$

The properties of the function $g(x)$ are obtained numerically. In Fig.1 is displayed the ratio of the new energy density and the habitual one, as a function of the variable T/T_{cr}.

Let us now give an estimate of the bound thermodynamics imposes on the deformation parameters. Consider the Helium whose specific heat at constant volume assumes the experimental values $12.4JK^{-1}mole^{-1}$ (The undeformed theory assigns the value 12.47 to any non relativistic gas). With the help of the appendix, we can compute the specific heat in the new theory $C_V = 12.47(1 + \sigma(\beta mkT))$. The measured value tells us that $12.39 < C_V < 12.59$. At first order, $\sigma(x) \sim x^2$. Assuming $T = 300^oK$, this gives $\beta \geq 10^{-45}$. The unit in which $beta$ is given is $s^2kg^{-2}m^-$. One then finds a minimal length uncertainty $\gamma \geq 10^{-16}$ meters which is quite close to the bound derived from atomic physics considerations [12]. Working with another inert gas(the argon), we end up with the same result. This value of β can now be used to obtain a bound on the relativistic characteristic temperature, $T_{cr} \geq 10^9K$. This is very far from the value $T_{cr} \sim 10^{16}K$ which corresponds to a minimal uncertainty in position of the order of the Planck scale. Taking the modified relations used here as a "fundamental theory" i.e. not a phenomenological approximation(coming from string theory for example) , one sees that a new physics may appear well before the Planck scale. Its specific effects will be seen essentially in the early universe but astrophysics may also provide a field of application since the temperatures involved can be near the characteristic relativistic temperature whose lower bound is 10^9K. The case $\beta = 0$ has been seen to have a specific heat which is constant (see $Eq.(2.29)$); the

argument we have used with the argon and the nitrogen to infer a bound on β in the case $\beta \not{0}$, can not be repeated here to constrain α. One can use the departure of the equation $Eq.(6.15)$ from the relation $PV = NkT$ to find the allowed zone $\alpha \leq 10^{(-3)}$ (the unit is m^{-2}) which is quite large .

III-A.3   The Einstein model for a solid.

When $\alpha = 0$, the spectrum of a harmonic oscillator is found to display a quadratic term [1] :

$$E_n = E_0 + an + bn^2 \tag{3.28}$$

with

$$E_0 = \frac{1}{2}\hbar\omega \left(\frac{1}{4\sqrt{r}} + \sqrt{1 + \frac{1}{16r}} \right) \tag{3.29}$$

$$a = \hbar\omega \left(\frac{1}{4\sqrt{r}} + \sqrt{1 + \frac{1}{16r}} \right) \tag{3.30}$$

$$b = \frac{\hbar\omega}{4\sqrt{r}} \tag{3.31}$$

where

$$r = \frac{1}{(2m\hbar\omega\beta)^2} \tag{3.32}$$

so that b vanishes when $\beta \to 0$. The free energy of a crystal with N such oscillators at its sites is

$$F = N(E_0 - kT \ln K_0(T, \beta)) \tag{3.33}$$

with

$$K_0(T, \beta) = \sum_{n=0}^{\infty} \exp\left(-\frac{an + bn^2}{kT} \right). \tag{3.34}$$

Its specific heat at constant volume reads

$$C_v = 3Nk \left[2T\frac{K_1(T, \beta)}{K_0(T, \beta)} + T^2\frac{K_2(T, \beta)}{K_0(T, \beta)} - T^2 \left(\frac{K_1(B, \beta)}{K_0(T, \beta)} \right)^2 \right] \tag{3.35}$$

with

$$K_1 = \frac{\partial K_0}{\partial T} \ and \ K_2 = \frac{\partial K_1}{\partial T}. \tag{3.36}$$

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A numerical treatment shows that the behavior of the specific heat in this model is very close to the one obtained in the undeformed case, for values of β in the range we found for the non interacting gas in the previous section. There appears to be a difference near the zero temperature but it is so tiny that it is experimentally undetectable .

III-B   The case $\beta = 0$.

In the new context, the SchröSchrodinger equation in the position representation leads to the energy spectrum

$$E_n = \frac{1}{2m}\left(\frac{2\pi\hbar\sqrt{\alpha}}{\arctan\sqrt{\alpha}L}n\right)^2 \tag{3.37}$$

for non relativistic non interacting particles in a uni dimensional box of length L. This leads to a measure which could be inferred from the uncertainty relation

$$\Delta q\Delta p \geq \frac{\hbar}{2}(1+\alpha q^2). \tag{3.38}$$

The partition function corresponding to the three dimensional case is

$$Z = \frac{1}{h^{3N}N!}\int\frac{d^{3N}\vec{p}d^{3N}\vec{q}}{(1+\alpha\vec{q}^2)^{3N}}\exp(-E/kT). \tag{3.39}$$

The measure of the phase space displayed in Eq.(3.39) lowers the influence of the large $\vec{q}^2$ i.e of the boundaries .

III-B.1   Non interacting non relativistic gas.

In this case the quantity J defined in Eq. (2.23) is found to be the volume integral

$$J = \frac{1}{V}\int\frac{d^3\vec{q}}{(1+\alpha\vec{q}^2)^3}. \tag{3.40}$$

As can be seen from Eq.(2.25), the fact that J has an explicit volume dependence breaks the relation $pV = NkT$. The last contributions to the entropy and the chemical potential vanish also because $J(\alpha, V)$ doesn't depend on the temperature and the particle number. The internal energy remains unchanged ($U_* = U$) and by way of consequence the specific heat at constant volume is also unmodified.

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Usually, the positions appear in the integrand of the partition function through interaction terms. The interaction between two particles depends on the vector separating them but this is not the case of Eq(3.12). The situation may look similar to that of a perfect gas in a constant gravitational field since the particles do not interact among themselves but couple to an external field. In this case, the partition function is the integral of the quantity $\exp\left(-\vec{p}^2 - mgz\right)/(kT)$. One sees that for the case , the factor $(1 + \alpha\vec{q}^2)$ lacks the proper kT piece to allow such a parallelism. *As a consequence, the thermodynamics generated by the K.M.M theory can not be interpreted as the usual one with a non trivial interaction.* We will see that the same consideration applies to the q oscillator in the last section.

The appearance of the norm of the position vector raises the question of the origin from which it is measured. This is related to the question of frame dependence which is not specific to the theories under study. In habitual statistical physics, the partition function involves the energy which is not the same for all observers, even galilean. The frame chosen is that of an observer at rest with respect to the container. In the case we are studying here, one has to specify a special point which will play the role of the origin. If for example we consider a spherical container, the center is a point with special status.

The integral displayed in Eq(3.40) is easily performed if we put the gas in a spherical container of radius R :

$$J = \frac{1}{V}\frac{\pi}{\alpha}\left[-\frac{R}{(1+\alpha R^2)^2} + \frac{1}{2}\frac{R}{1+\alpha R^2} + \frac{1}{2\sqrt{\alpha}}\arctan(\sqrt{\alpha}R)\right]. \qquad (3.41)$$

The Eqs. , (2.25), (2.26), (2.27)show that the pressure, the entropy and the chemical potential are all affected by the new commutation relations. Let us, for simplicity, study two limiting cases : the small and the huge volumes.

What happens when the volume can not be neglected in front of $\alpha^{-(3/2)}$? The expression of J becomes

$$J = \frac{1}{\alpha^{3/2}V}\left[a - b\frac{1}{\alpha^{3/2}V} + c\frac{1}{\alpha^{5/2}V^{5/3}}\right] \qquad (3.42)$$

with

$$a = 2.4674, b = 17.546, c = 78.6488. \qquad (3.43)$$

Retaining the first two contributions, the pressure takes the form

$$P = NkT\left(\frac{b}{a}\frac{1}{\alpha^{3/2}}\frac{1}{V^2} - \frac{11}{3}\frac{c}{a}\frac{1}{\alpha^{5/2}}\frac{1}{V^{2/3}}\right). \qquad (3.44)$$

This theory predicts an important change in the thermodynamic properties for huge volumes; the correction to the pressure for small volumes given in Eq. (6.15) in the appendix is negligible. But the last expression shows that for very big containers, the pressure of a non relativistic gas may become negative. However, this does not contradict any experimental bound since for typical values of α like the ones of [12], [15] , one would need a recipient whose volume is greater than the earth . A negative pressure is a priori puzzling because it does not appear in everyday thermodynamics. But one knows it is an essential ingredient of the inflation scenario in cosmology [17]. We will come back to this point in the conclusions.

Similarly, one finds for the entropy and the chemical potential in the case of important volumes :

$$S = Nk\left\{\frac{5}{2} + \ln\left\{\left(\frac{a}{\alpha^{3/2}}\right)\frac{1}{N}\left(\frac{2\pi mkT}{h^2}\right)^{3/2}\right\}\right\} \qquad (3.45)$$

$$\mu = -kT\left\{\ln\left\{\left(\frac{a}{\alpha^{3/2}}\right)\frac{1}{N}\left(\frac{2\pi mkT}{h^2}\right)^{3/2}\right\}\right\}. \qquad (3.46)$$

The formula giving the entropy in big containers does not involve the volume itself and tells us that an adiabatic expansion is nothing else than a isotherm. The expression of the chemical potential shows that when the reaction

$$aA + bB \longrightarrow cC + dD \qquad (3.47)$$

attains equilibrium, the following relation holds between the number of particles N_i of the species i

$$\frac{N_C^c N_D^d}{N_A^a N_B^b} = \tau T^{3/2(c+d-a-b)} \qquad (3.48)$$

with τ a quantity which does not depend on the temperature or the pressure. This is very different from the action of masses law in which only the concentrations are constrained by a similar relation.

We now investigate entropy mixing. Consider two gases A, B contained in two compartments (of respective volumes V_A, V_B of a container separated by an impermeable wall). They are at the same temperature and pressure, their respective numbers of particles are N_A, N_B .Now, let us remove the wall. In the non deformed theory$(\alpha = \beta = 0)$, the temperature and the

the pressure remain the same (due to the conservation of energy and the equation of state). According to Eq.(3.22), the increase of entropy for this process is non vanishing.

In the theory we are studying, the equation of state specified by Eq.(3.44) and the internal energy lead to the same pressure and temperature after mixing. However, the expression of the entropy is now given by Eq.(3.45) . For huge volumes, Eq.(3.22) gives that ΔS vanishes. This differs significantly from the non deformed theory. Let us consider two important but equal volumes $V_A = V_B = V$ and two equal number of particles $N_A = N_B = N$ of two different gases. The habitual theory states that the entropy mixing is $2Nk \log 2$ while this K.M.M deformation predicts $\Delta S = 0$.

III-B.2 The ultra relativistic non interacting gas.

J is found to be exactly the same as in the previous section so that compared to the undeformed case, the state equation, the entropy and the chemical potential change while the specific heat at constant volume remains the same.

$$S = Nk \left\{ 4 + \ln \left[\left(\frac{a}{\alpha^{3/2}} \right) \frac{1}{N} \left(\frac{(8\pi)^{1/3}hc}{kT} \right)^3 \right] \right\} \tag{3.49}$$

$$\mu = -kT \ln \left\{ \left(\frac{a}{\alpha^{3/2}} \right) \frac{1}{N} \left(\frac{(8\pi)^{1/3}hc}{kT} \right)^{3/4} \right\} \tag{3.50}$$

The absolute value of the chemical potential is lowered compared to the habitual case. The formula Eq(3.25) is still valid, but with a different τ. In summary, above the critical volume $\alpha^{-3/2}$, the equation of state takes the form

$$\frac{PV}{NkT} = 1 + f(V/V_{cr}) \qquad \rho = 3p(1 + f(V/V_{cr})) \tag{3.51}$$

for a relativistic gas; in the non relativistic case, the factor 3 in the last equation should be divided by 2. The function $f(x)$ is plotted in Fig.2 . It begins at zero for a vanishing x and quickly hugs to the horizontal asymptot $y = -1$. In this model, the parameter α is fixed once and for all. If one considers an increasing volume for a fixed quantity of gas(fixed N) and a fixed temperature(fixed T), the last two equations show that ρ/p and p/T go to zero.

An important issue for gases presenting a non trivial equation of state is the possibility of phase transitions. For example, a Van der Waals gas

exhibits a condensation plateau when it is compressed at (fixed temperature)under a critical volume (obtained by Maxwell construction). In the phase diagram , this is linked to the existence of a critical point where the two first derivatives of the pressure with respect to the volume(on the appropriate isothermal curve) are vanishing. From Eq.(3.30) one can isolate P and see that

$$\frac{\partial P}{\partial V} = \frac{NkT}{V^2} f_1(V/V_{cr}) \qquad \frac{\partial^2 P}{\partial V^2} = \frac{NkT}{V^3} f_2(V/V_{cr}) \qquad (3.52)$$

$$f_1(x) = -1 - f(x) + x f'(x) \qquad f_2(x) = 2 + 2f(x) - 2x f'(x) + x^2 f''(x) \quad (3.53)$$

The functions $f_1(x)$ and $f_2(x)$ are plotted in Fig.3 an Fig.4 respectively. They show that there is no such a phase transition here.

What we have done for gases is similar to Maxwell Boltzmann statistics because we did not symmetrize the wave functions for the bosons nor antisymmetrize for the fermions. At the fundamental level, one can ask if the concept of spin is relevant in these theories and, in the case the answer is positive, one may wonder about the spin-statistics theorem in the new context.

The commutation relations given in Eq. (3.1) and Eq.(3.2) are not invariant under a Lorentz transformation. As the spin of a particle is defined through the behavior of its wave function under this group, one apparently looses the notion of spin here. However, this is not obligatory true. For example, taking the ultimate structure of space-time to be a particular commutative geometry, the relevant group of isometries is not the Poincarré one but a q deformation of it. There is a new algebra but a spin has been defined in these theories and the wave functions for particles of spin $1/2$, 1, etc... have been found [18]. Although the question has not been addressed in K.M.M. theory, we expect a similar situation to occur. Considering as an illustration the replacement of the Dirac equation by

$$(\gamma^\mu \hat{p}_\mu + m)\psi = 0 \qquad (3.54)$$

with $\hat{p}_\mu$ given by the generalization in three dimensions of Eq. (3.9), the relevant symmetries of this equation should be linked to an inhomogeneous algebra since the operator is non-linear. We expect a generalization of the notion of spin which may conserve the spin-statistic theorem. This is not really crucial here since we are performing a Boltzmann like construction

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which is known to be valid at high temperature and does not discriminate between bosons and fermions. What we called " relativistic systems" in the deformed theories are nothing than systems described by the habitual relativistic equations, but with the momenta replaced by their quasi position expression in the model where α vanishes, for example. This can in principle be made rigorous using the functional integral introduced in [3] since in this formalism one does not need to introduce neither the commutation relation nor the scalar product from the start.

The theories studied here are unusual in many respects. we have seen that a minimal length uncertainty can exist for specific parameters. The fact that the commutator of two quantum operators is not simply proportional to the Poisson bracket of their classical counterparts creates a situation in which a quantity conserved at the classical level may not be conserved at the quantum level. This implies that one has to be careful when studying ensembles more complicated than the canonical one.

We studied a system which exchanges only energy with the surroundings. When the number of particles is not fixed, one has to work in the grand canonical ensemble. However, it is known in usual thermodynamics that the two descriptions yield essentially the same predictions(equation of state, specific heat, $\cdots$). A real difference appears essentially when describing phase transitions.

Our treatment concerned only non interacting gases but we know they only approximate the real ones when the density is small. We think it is physically irrelevant to include interactions for habitual gases like hydrogen, ammonia, nitrogen, etc $\cdots$. Taking for example the theory where only $\beta \neq 0$, we saw that a new behavior sets in at very high temperature. But at this energy scale, nitrogen does not exist anymore and the use of an interaction of the Lennar-Jones type is not justified; at very high temperatures, the existing species are elementary particles like photons, neutrino, leptons, quarks $\cdots$. One may expect to have a different situation for the model in which only α is non vanishing ; one may conceive that the expansion of the universe may provide the necessary huge volumes. However, as the universe gets bigger its density decreases and the mean distance between the particles becomes too important ;they nearly don't feel the interaction.

IV The q deformed oscillator.

A space with non commutative coordinates is called a quantum plane; its covariant differential calculus has been investigated [19]. A quantum group can be understood as a set of linear transformations acting on it. The commutation relations compatible with this structure [20] are quickly summarized below.

The momentum and position operators P, X satisfy the commutation relation

$$q^{1/2}XP - q^{-1/2}PX = iU. \tag{4.1}$$

The deformation parameter q is strictly greater than one; the unitary element U has been introduced to allow hermitian momentum and position :

$$P^+ = P \ \ X^+ = X \ \ U^+ = U^{-1}. \tag{4.2}$$

It is also subject to the conditions

$$UP = qPU \ , \ UX = q^{-1}XU. \tag{4.3}$$

Introducing the extra parameter M , one defines the operators

$$a^+ = \bar{\alpha}U^{2M} + \bar{\beta}PU^M \tag{4.4}$$

$$a = \alpha U^{2M} + \beta U^{-M}P \tag{4.5}$$

and finds that they obey a modified Heisenberg algebra

$$aa^+ - q^{-2M}a^+a = 1. \tag{4.6}$$

We first investigate what changes in the Einstein model of a solid . In this theory, the harmonic oscillator has been defined [16] as having the hamiltonian

$$H = \omega a^+a \tag{4.7}$$

with a^+, a the operators defined in Eqs.(4.4),(4.5) and whose algebra is given by Eq. (4.6).

A special role is played by the eigenvalue

$$E_\infty = \frac{\omega}{1 - q^{-2M}} \tag{4.8}$$

of the hamiltonian :

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a) There is an infinite subset of the spectrum which is bounded from above by $E_{[\infty]}$:

$$E_m^* = \omega \frac{1 - q^{-2Mm}}{1 - q^{-2M}} \leq E_{[\infty]} \qquad m = 0, 1, 2, \cdots. \qquad (4.9)$$

For these states, a raises the energy while a^+ lowers it.

b) The remaining part of the spectrum

$$E_n = \omega \left(\frac{1}{1 - q^{2M}} + \beta\bar{\beta}q^{2n} \right) \quad n = -\infty, \cdots, +\infty \qquad (4.10)$$

(with β depending on the mass m and the frequency ω) faces the reverse, common situation with a^+ playing the role of a creation operator.

c) For $E = E[\infty]$, a and a^+ do not change the eigenvalue.

To give an example, the operators

$$\hat{P} = -i\frac{\partial}{\partial x} + \frac{1}{\sqrt{1 - q^{-2M}}}\gamma \ , \quad \hat{X} = x \qquad (4.11)$$

$$U = q^{-\frac{i}{2}(\hat{X}\hat{P} + \hat{P}\hat{X})} \qquad (4.12)$$

are found to obey the relations given in Eqs. (4.1-4.3) which define the q deformed phase space. Introducing the operators a, a^+ as specified by Eq. (4.4-4.5) and expanding the hamiltonian $\omega a^+ a$, one finds it consists of two terms. The first part is the hamiltonian of a free oscillator and the remaining part can be seen as an interacting term 0 as a series in $\sqrt{h}$ with $h = e^q - 1$.

The link between the parameters α, β, γ on one hand and the mass, the frequency of the unperturbed oscillator on the other hand are

$$\alpha = \pm\frac{i}{\sqrt{2Mh}} \qquad \gamma = \pm\sqrt{2m\omega} \qquad \beta = \frac{i}{\sqrt{2m\omega}}. \qquad (4.13)$$

It should be emphasized that the K.M.M. oscillator presents a spectrum which reduces to the habitual one as the deformation parameter is sent to zero. This is evident from Eqs. (3.29) and (3.30). The situation of the q algebra is different and so one expects it should lead to an entirely new

thermodynamics, even for a small value of the deformation parameter. Let us study a crystal whose sites are occupied by q oscillators.

Constructing the partition function

$$Z(T,V,1) = \sum_{m=0}^{\infty} e^{-\frac{E_m^*}{kT}} + \sum_{n=-\infty}^{+\infty} e^{-\frac{E_n}{kT}} \tag{4.14}$$

one finds that the first sum diverges since $E_m^* < E_{[\infty]}$. How can we handle this? The extreme reaction would be to disregard the all theory as unphysical. Another possibility may be a renormalization of the partition function but this is questionable since we are not doing quantum field theory. Another point of view is that these states may be interpreted as spurious and eliminated. We do not have a clear option between these possibilities for the moment. Introducing $\sigma = \sqrt{\omega\beta\bar{\beta}q}$ one rewrites the remaining part as

$$Z(T,V,N) = e^{-\frac{\omega}{(1-q^2)kT}} \sum_{n=-\infty}^{+\infty} e^{-\frac{\sigma^n}{kT}}. \tag{4.15}$$

If $\sigma < 1$, one has

$$\lim_{n\to+\infty} \exp(-\sigma^n) = 1 \tag{4.16}$$

so that $Z(T,V,1)$ still diverges. The last option would lead us to drop also the states corresponding to high positive values of n. The question would then be to fix the index where the truncature should take place.

If we discard the E_m^*, we were left with $E_{[\infty]}$ as the lowest state. As it corresponds to E_n for $n = 0$, we think it should be kept in the physical spectrum. So, our truncature begins at $n = 1$ in this case :

$$Z(T,V,1) = e^{-\frac{\omega}{(1-q^2)kT}} \sum_{n=-\infty}^{0} . e^{-\frac{\sigma^n}{kT}} \tag{4.17}$$

When $\sigma > 1$, the situation is reversed :

$$Z(T,V,1) = e^{-\frac{\omega}{(1-q^2)kT}} \sum_{n=0}^{+\infty} e^{-\sigma\frac{q^n}{kT}}. \tag{4.18}$$

Writing $q = 1 + h$ and retaining the first two terms of the binomial formula to approximate q^n, one deals with a partition function similar to the one used in the K.M.M model. The same results apply, provided one chooses a value of h in the range which gives to β the values compatible with the bound given in section 3. Discarding the extra states has one nice feature: the thermodynamics is smooth in the deformation parameter.

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V   Conclusions

Our investigation of the thermodynamics implied by two special K.M.M. deformations of the commutation relations have shown that a new physics appears at high temperatures in one case and for huge volumes in the other one. In the two cases, the universe should provide the best laboratory since the modified commutators can produce specific, visible signatures . In fact, the evolution of the universe will be affected in more than one aspect. Of course, the new equation of state of the radiation gas which fills the early universe influences the evolution of the scale factor through Einstein equations. In addition, the new expressions of the entropy change the form of the adiabatic processes. As in standard cosmology the expansion of the universe is thought to be isentropic, this will enter into play. The fact that chemical potentials receive contribution corrections will affect the densities at equilibrium . Finally, the theory exhibiting a minimal length uncertainty may forbid the singularity present in the standard big bang scenario. A similar reasoning was advocated to argue that the Hawking evaporation of a black hole may halt without using the complementarity hypothesis [6].

Let us say a few words about the negative pressures found in section3.3 . In the inflation scenario, one uses a scalar field whose classical time evolution generates such a non positive pressure for a given period, because of a well chosen potential [17]. Eq(3.21) suggests that one may achieve such pressures with ordinary matter, provided the commutation relations are modified. But to achieve inflation, one needs the relation $p < -\rho/3$ and this not obvious from Eq.(3.21). One also needs a mechanism to return to normal matter i.e positive pressures. Certain mechanisms of inflation are inefficient because their period of exponential growth never ends. This is the case in topological inflation where the core of a monopole, a cosmic string or a domain wall expands endlessly. It should also be stressed that here the negative pressures appear only when the volume gets big enough, contrary to standard inflation where it happens very early in the course of the universe.But the true challenge here, which may rule out all the model corresponding to the case $\beta = 0, \alpha \neq 0$ is that there is no evidence for a preferred origin. A straightforward possibility is that this theory may be interpreted as imposing a maximal physical volume since going above it one gets into trouble. One of the most important novelties is the behavior of the entropy mixing in the $\alpha = 0$ case which seems to spoil the interpretation of the entropy derived from Eq. (2.11) as a variable counting the number of states. The breaking

of homogeneity makes it likely that one should consider Tolman (position dependent) temperatures like in curved space times. But in such spaces the fact that the quantity $T\sqrt{-g_{00}}$ is constant is a vital information which seems to have no counterpart in the theory $\alpha \neq 0$. This model is also useless in cosmology since the existence of an origin is in contradiction with the homogeneity hypothesis. One may use all the undesirable consequences of this the model to argue it is unphysical, like the quantum theory on a Moyal plane modifying the time-position commutator $[x_o, x_i]$ has been discarded because it leads to a non unitary theory.

We plan to analyze the cosmological incidences of the model $\alpha = 0$ in a near future. The influence of the new physics in the early universe is at the center of a recent work using Unruh and Jacobson modified dispersion relations [21]. Our approach will differ by the fact the dispersion relation will come from an assumption on the quantum space time . The commutation relations studied here can be interpreted as phenomenological consequences of string or M theory [22, 23]. This paper suggests that string cosmology is not uniquely characterized by the evolution of the fields which appear at the lowest order i.e. dilaton, $\cdots$ but also by some non trivial statistical effects. [17].

We have restricted ourselves to the two extreme cases $\alpha = 0$ and $\beta = 0$. One can ask what happens when α and β are simultaneously non vanishing. An uncertainty relation similar to the one displayed in Eq. (3.4) exists but one does not possess a position nor a momentum representation. To deduce classical statistical physics from quantum statistical physics as we did , one has to deal with the Bargmann representation [3] which is more complicated. In this case , J will depend on both T and V. The reserves we expressed due to the presence of a preferred origin also apply here.

Can a similar treatment be performed for gases in the q deformed algebra? The situation may prove difficult since for a free particle in this theory, the momentum is quantized without advocating boundary conditions [10].

VI   Appendix

VI-A The Non relativistic non interacting gas in the $\alpha = 0$ theory

The change of variable $x = \vec{p}^2/2mkT$, allows one to write J as the dimensionless integral

$$J = 2\pi^{-1/2} \int_0^\infty dx e^{-x} x^{1/2} (1 + 2\beta mkTx)^{-3}. \qquad (6.1)$$

This integral can not be performed analytically. Nevertheless a useful approximation can be obtained which gives a good qualitative picture. Being the product of an increasing and a decreasing exponential, the integrand presents a maximum at

$$\bar{x} = \frac{-(1 + 5\beta mkT) + \sqrt{1 + 14\beta mkT + 25\beta^2 m^2 k^2 T^2}}{4\beta mkT}. \qquad (6.2)$$

Calling $\phi(x)$ the integrand in Eq.(3.34) , a saddle point approximation gives

$$J = (\phi(\bar{x}))^{3/2} \frac{\sqrt{2\pi}}{\sqrt{-\phi''(\bar{x})}}. \qquad (6.3)$$

The fact that $\phi(x)$ is maximal at $\bar{x}$ simplifies the expression of the second derivative

$$\phi''(\bar{x}) = (1 + 4\beta mkT\bar{x})(4\beta mkT\bar{x}^2 - 4\bar{x} + 1). \qquad (6.4)$$

In this case J is a function of the temperature but it does not depend on the particle number or the volume. Consequently the equation of state $PV = NkT$ is still valid here but the molar specific heat at fixed volume is no more the constant $3R/2$. Let us compute the first correction to J. A gas presents a non relativistic behavior at low temperature; in this limit we obtain, introducing the particle dependent characteristic temperature $T_c = 1/\beta mk$ and find to the first order in T_c/T

$$\bar{x} = \frac{1}{2} - 3\frac{T}{T_c} \qquad (6.5)$$

$$J = \sigma_0 + \sum_l \sigma_l (T/T_c)^l \qquad (6.6)$$

with

$$\sigma_0 = 1.20353, \sigma_1 = -10.8318, \sigma_2 = 93.8755, \cdots. \tag{6.7}$$

Now

$$S = Nk \left[\frac{5}{2} + \ln \sigma_1 + \ln \left\{ \frac{V}{N} \left(\frac{2\pi mkT}{h^2} \right)^{3/2} \right\} + 2\frac{\sigma_1}{\sigma_0} \left(\frac{T_c}{T} \right) \right] \tag{6.8}$$

$$\mu = -kT \left[-\ln \sigma_0 + \ln \left\{ \frac{V}{N} \left(\frac{2\pi mkT}{h^2} \right)^{3/2} \right\} + \frac{\sigma_1}{\sigma_0} \frac{T}{T_c} \right] \tag{6.9}$$

and the equation of state is

$$\rho = P \left(\frac{3}{2} - \frac{\sigma_1}{\sigma_0} \frac{T}{T_c}. \right) \tag{6.10}$$

An adiabatic process takes the form

$$V = c^{te} N T^{-3/2} e^{\frac{2\sigma_1}{\sigma_0} \frac{T}{T_c}} \tag{6.11}$$

while densities $X_i = N_i/V$ at equilibrium for the reaction displayed in Eq.(3.24) obey

$$\frac{X_C^c X_D^d}{X_A^a X_B^b} = c^{te} T^{3/2(-a-b+c+d)} e^{\left(-\frac{\sigma_1}{\sigma_0} \left(\frac{a}{T_c^A} + \frac{b}{T_c^B} + \frac{c}{T_c^C} + \frac{d}{T_c^D} \right) \right)} \tag{6.12}$$

since each particle, having its own mass, possesses a specific critical temperature.

VI-B   The Non relativistic non interacting gas in the $\beta = 0$ theory

If one considers a small container, then the following approximation is accurate for the quantity J :

$$J = 1 - a_1 \alpha V^{2/3} \tag{6.13}$$

with

$$a_1 = \frac{4}{5} \left(\frac{3}{4} \right)^{5/3} \pi^{-2/3}. \tag{6.14}$$

The new pressure

$$P = NkTV^{-1} \left(1 - \frac{2}{3} a_1 \alpha V^{2/3} \right) \tag{6.15}$$

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is lower than the one obtained for the same gas in the unmodified algebra. The minus sign suggests that the pressure may become negative. However, as α must be tiny, this would occur for huge volumes but then the approximation specified in Eq. (3.14) is no more valid.

The entropy and the chemical potential become

$$S = Nk \left[\frac{5}{2} - a_1 \alpha V^{2/3} + \ln \left\{ \frac{V}{N} \left(\frac{2\pi mkT}{h^2} \right)^{3/2} \right\} \right] \qquad (6.16)$$

$$\mu = -kT \left[\ln \left\{ \frac{V}{N} \left(\frac{2\pi mkT}{h^2} \right)^{3/2} \right\} + a_1 \alpha V^{2/3} \right] \qquad (6.17)$$

while the specific heat remains the same. One sees that for small volumes nothing really new appears.

VII   figure captions

- Fig1 The quantity $1 + g(T/T_{cr})$ shows that the energy density due to a radiation of fixed pressure decreases at high temperatures in the $\alpha = 0$ case.

- Fig2 The function $1 + f(x)$ which modifies the equation of state when $\beta = 0$ is depicted. It tends quickly to its asymptotic value. For a system with fixed T, N, the pressure falls more quickly when the volume is increased in comparison to the undeformed case.

- Fig3 The function $f_1(x)$ linked to the first derivative of the pressure does not vanish for finite volumes so that the isotherms in the $\beta = 0$ theory do not have an extremum in the P, V diagram.

- Fig4 The function $f_2(x)$ linked to the second derivative of the pressure does not vanish neither. This and the previous figure proves there is no phase transition.

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0.75
0.7
0.65
0.6
0.55

20   40   60   80   100